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A METHOD TO ESTIMATE THE PRIMORDIAL POWER SPECTRUM FROM COSMIC MICROWAVE BACKGROUND DATA

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ABSTRACT

A precise determination of the primordial spectrum of matter density fluctuations at superhorizon scales is essential to understanding large-scale structure in the universe. Attempts to constrain or obtain the primordial spectrum using data on cosmic microwave background (CMB) anisotropies have relied on statistical and correlation analyses that assume a power-law spectrum. We propose a method to derive $P(k)$ directly from the CMB angular power spectrum that does not presuppose the need to know anything about its functional form. The method consists of a direct inversion of the Sachs-Wolfe formula. Using this new analysis technique and *COBE* data, we obtain an empirical $P(k)$ that (1) supports a power-law parameterization and (2) has an amplitude and spectral index consistent with previous analyses of the same data. We obtain for the spectral index $n = 1.52 \pm 0.4$ when the second-year *COBE* data are used and $n = 1.22 \pm 0.3$ using the 4 year data set.

Subject headings: cosmic microwave background — cosmology: theory — large-scale structure of universe

1. INTRODUCTION

The mechanism for large-scale structure formation in the universe calls for primordial density fluctuations (PDFs) in the early universe. A knowledge of the spectrum of PDFs, $P(k)$, would allow one to compute the rms mass fluctuation on a given scale, $\delta M/M$, and the peculiar velocity field. Inflation predicts a scale-invariant spectrum $P(k) \propto k$ (Harrison 1970; Zeldovich 1972).

The availability of cosmic microwave background (CMB) data at large angular scales (from *COBE* [Bennett et al. 1996, 1994], Tenerife [Hancock et al. 1994], and FIRS [Ganga et al. 1993]) has made it possible at least in principle to probe the shape of the PDF spectrum. Theoretical uncertainties (i.e., "cosmic variance") and experimental constraints such as sampling variance, low signal-to-noise ratio, and Galactic contamination, however, impose very stringent limitations on the ability to obtain the original spectrum. In order to deal with the effects of an equatorial cut in the portion of the celestial sphere dominated by diffuse Galactic emission, Górski et al. (1994) have found a new orthogonal set of basis functions to represent the scalar radiation field. An alternative used by Wright et al. (1994a, hereafter WRI94) applies weights to the spherical harmonic decomposition of $\Delta T/T$ in order to correct for aliasing among different l-terms that result in the cut sphere when the monopole and dipole terms are removed.

Most of the analyses of CMB data aimed at probing $P(k)$, however, have been made using a statistical maximum likelihood analysis on the angular power spectrum or the autocorrelation function under the assumption of a power law for $P(k)$. In view of the aforementioned intrinsic and instrumental limitations, the need for new and alternative methods of analysis is well justified. We propose a new technique to obtain $P(k)$ directly from the CMB angular power spectrum that *does not assume any particular form for* $P(k)$, thus allowing one to test for deviations from power-law models. The method is based on a direct integration of the Sachs-Wolfe formula (Sachs & Wolfe 1967) for the angular spectrum coefficients. It is shown that by a straightforward application of the mean value theorem the Sachs-Wolfe integral can be inverted, resulting in a robust estimate of $P(k)$ over the wavelength range available to CMB experiments.

2. ALGORITHM

Expressing the CMB temperature anisotropies in the usual spherical harmonic expansion allows one to define the angular power spectrum, C_l:

$$\frac{\Delta T}{T} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) \tag{1}$$

(Bond & Efstathiou 1987), with

$$C_l \equiv \langle | a_{lm} |^2 \rangle . \tag{2}$$

The C_l used here are related to the rotationally invariant rms multipole moments used by *COBE* according to $\Delta T_l^2 = (2l + 1)T_0^2 C_l/(4\pi)$, with T_0 the monopole temperature. Experiments provide estimates for C_l, and their connection with theory is established by means of the Sachs-Wolfe effect. For large angular scales, the C_l are

$$C_l = \left(\frac{4\pi}{5}\right)^2 \int_0^\infty P(k) \left[\frac{j_l(k)}{k}\right]^2 dk \tag{3}$$

(Fabbri, Lucchin, & Matarrese 1987), where we have made k nondimensional ($k \leftarrow 2ck/H_0$) and the formula is valid for $\Omega_0 = 1$.

We attempt to obtain straightforward information about the spectrum of PDFs by a direct inversion of equation (3).

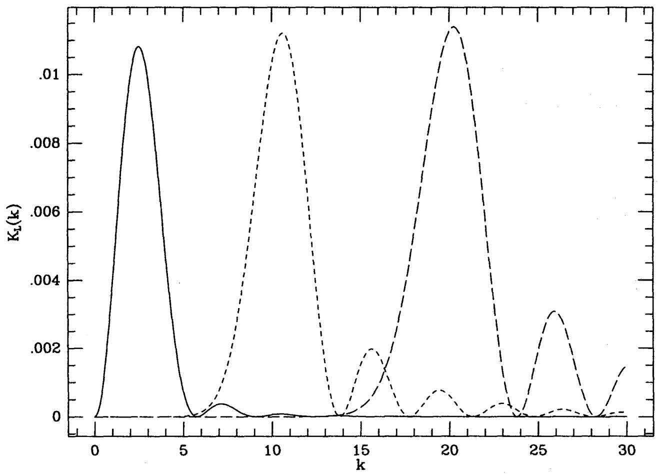

FIG. 1.—Kernel function $K_l(k)$ for $l = 2$ (*solid line*), $l = 9$ multiplied by 100 (*short-dashed line*), and $l = 18$ multiplied by 1000 (*long-dashed line*).

The primary data are the angular correlation coefficients measured by experiments. For this purpose, we have devised a method that, in addition to its simplicity, has two basic virtues: (1) it makes a minimum of assumptions about the form of the spectrum, and (2) it can readily be extended to include any new information on the C_l that might come from future experiments.

Let us introduce the integral

$$I_l = \int_0^\infty \left[\frac{j_l(k)}{k}\right]^2 dk\,, \quad l \geq 2\,. \tag{4}$$

Note that every factor in the integrand of equation (3) is either positive definite or at least nonnegative and smooth. This allows us to establish the following identity:

$$P(\bar{k}_l) = \frac{1}{I_l}\int_0^\infty P(k)\left[\frac{j_l(k)}{k}\right]^2 dk = \frac{C_l}{(4\pi/5)^2 I_l}\,, \tag{5}$$

which is an application of the mean value theorem (Stromberg 1981, p. 328). Equation (5) simply states that, for some value of its argument, here denoted as $\bar{k}_l$, the value of $P(k)$ has to match the right-hand side of equation (5). This is true under the conditions stated above. Then the evaluation of equation (4) and the knowledge of the C_l for a given set of values of l yield the value of the left-hand side of equation (5). In principle, it can be seen that the Sachs-Wolfe formula combines all scales in k-space in the coefficients C_l. The reason why the direct-inversion prescription works can be seen by examining the form of the kernel, $K_l(k) \equiv [j_l(k)/k]^2$, in equation (5). The spherical Bessel functions are quasi-periodic with decreasing amplitude for large k; thus the kernel for each l is also a periodic function, but with its first peak being the only dominant contribution (see Fig. 1). The maximum of the peak is located approximately at $k \approx 0.7 + 1.1l$ (with the dimensionless k used here); thus, for each l in the kernel, one is probing a well-defined and independent scale.

Now we need to estimate with enough accuracy the arguments $\bar{k}_l$ in order to complete the table of $P(k)$ versus k. A way in which this can be accomplished is by recycling, in a new light, Zeldovich's recourse of estimating functions through the use of power laws, whenever the range of values of k of physical interest is small (only very large scales, in our case). Therefore, appealing to the same kind of reasoning, we proceed to obtain the values of k_l by means of the estimative relation

$$(\bar{k}_l)^s \approx \frac{1}{I_l}\int_0^\infty k^s\left[\frac{j_l(k)}{k}\right]^2 dk\,, \tag{6}$$

where s is any "reasonable" value, which must be within a certain range that satisfies the criteria of convergence and physical plausibility. Of course, the values of $\bar{k}_l$ obtained in this fashion cannot be very sensitive to the particular s chosen as estimator in equation (6). In fact, the use of k^s as the estimator function is not necessary; it is only one of the simplest that will do the job. Other, more elaborate estimators could be used, but as will be seen, this is not necessary.

We have evaluated equation (6) for the ranges $0.5 \leq s \leq 1.5$ and $2 \leq l \leq 12$, with the results shown in Table 1. As can be quickly noted, the maximum effect caused by varying s occurs for small l, with a spread of around $\pm 7\%$ in the worst case. For larger l, these variations grow smaller and their effect is not of importance.

TABLE 1

VARIATION OF ESTIMATED ARGUMENTS $\bar{k}_l$ WITH s

l	$\bar{k}_l(s = 1.0)$[a]	$\Delta\bar{k}_l(s = 1.5)$	$\Delta\bar{k}_l(s = 0.5)$
2	2.782	+0.18	−0.16
3	4.168	+0.20	−0.17
4	5.494	+0.23	−0.18
5	6.790	+0.26	−0.20
6	8.067	+0.28	−0.22
7	9.329	+0.30	−0.24
8	10.574	+0.31	−0.25
9	11.815	+0.33	−0.27
10	13.029	+0.33	−0.28
11	14.249	+0.34	−0.29
12	15.437	+0.34	−0.29

[a] The central values of the $\bar{k}_l$ are taken for the $s = 1.0$ case. The last two columns are the deviations in $\bar{k}_l$ from the central value when $s = 1.5$ and 0.5 are used, respectively.

3. ANALYSIS AND CONCLUSIONS

Using the Hauser-Peebles angular power estimator, WRI94 obtained values for the T_l^2-coefficients that are a linear combination of the C_l (see Table 1 in WRI94). A numerical integration of the inversion formula with $s = 1.0$ and the angular power spectrum in the range $3 \leq l \leq 18$ derived from WRI94 was used to obtain the $P(k)$-points illustrated in Figure 2. The error bars come from the uncertainties in the *COBE* data for the C_l. Coefficients beyond $l = 18$ were not included because the angular power spectrum seen by *COBE* for those values of l is dominated by noise and the associated angular scales are beyond *COBE*'s angular resolution. The observed angular power spectrum, $C_l^{\rm obs}$, is related to the theoretical spectrum by $C_l^{\rm obs} = G_l^2 C_l$, where G_l is the beam profile in terms of the coefficients in an expansion in Legendre polynomials. We have used the G_l in Wright et al. (1994b). The C_l are obtained by inverting the $T_{l,l'}$ matrix in Table 1 of WRI94. This is a matrix of dimensions $l_{\rm max} \times \infty$, which formally does not have an inverse. However, for the particular case under consideration, an

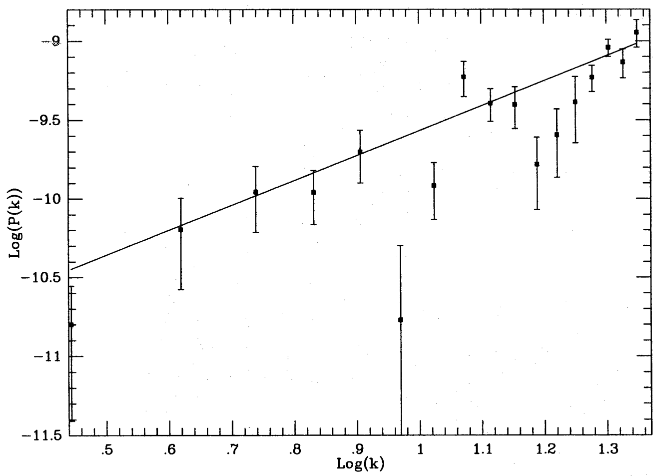

FIG. 2.—Spectrum of PDFs as derived from *COBE*'s 2 year angular power spectrum and best fit to a power-law function.

approximate inverse can be computed by noting that the nonzero matrix elements away from the diagonal follow a scaling law, $T_{ij} \propto |i-j|^{-2.12}$. The expansion of the C_l in terms of the T_l is truncated at a point where additional terms contribute a negligible amount relative to the measurement errors. Figure 3 shows the resulting C_l.

One can test the power-law *Ansatz* by attempting to fit $P(k)$ to a function of the form $P(k) \propto Q^2 k^n$. Here Q denotes the rms quadrupole normalization, more commonly written as $Q_{\text{rms-ps}}$. A maximum likelihood method taking into account the full covariance matrix was used. To find the model-dependent covariance matrix, $\mathbf{M}(Q, n)$, a Monte Carlo procedure was followed: First, the model parameters n and Q were fixed to generate realizations of the CMB angular power spectrum. These realizations of the C_l-coefficients follow a χ^2 distribution with $2l+1$ degrees of freedom and have mean values given by equation (4.18) of Bond & Efstathiou (1987). For each C_l-realization, our inversion method delivers a corresponding $P(k)$. The average of these $P(k)$'s over the ensemble of N_r realizations,

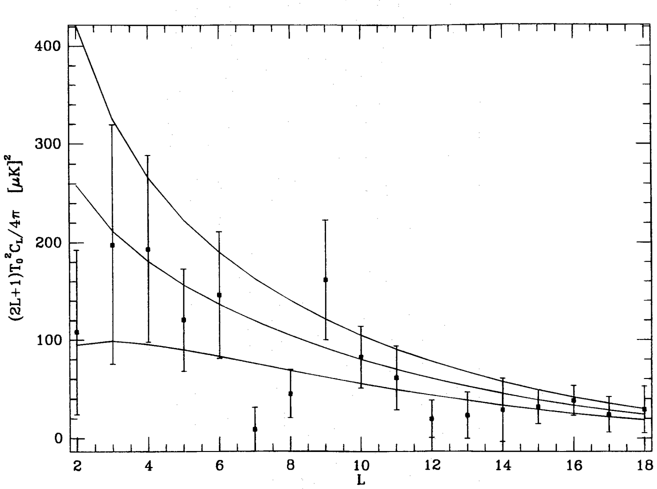

FIG. 3.—*COBE* 2 year angular power spectrum (*squares*) compared to the power spectrum corresponding to our best power-law fit of $P(k)$ in Fig. 2. The external lines define the $\pm 1\ \sigma$ band expected from cosmic variance.

$\langle P(k) \rangle$, was computed, as well as the covariance matrix:

$$M_{ij}(Q, n) = \frac{1}{N_r - 1} \times \sum_m^{N_r} [P_m(k_i) - \langle P(k_i) \rangle][P_m(k_j) - \langle P(k_j) \rangle] \,. \quad (7)$$

Finally, the likelihood function $L(Q, n)$ was computed:

$$-2 \ln L(Q, n) = \boldsymbol{d}^T \mathbf{M}^{-1}(Q, n) \boldsymbol{d} + \ln \det \mathbf{M}(Q, n) + \text{const} \,, \quad (8)$$

where the deviation vector $\boldsymbol{d}$ is the difference between a data point and the corresponding theoretical mean value from Monte Carlo realizations, $d_i = P(k_i) - \langle P(k_i) \rangle$. The covariance matrix is normalized so that the second term of equation (8) equals the χ^2-statistic. This procedure was repeated for several values of Q and n, forming a discrete sampling of $L(Q, n)$ inside a grid defined by the ranges $n = 0.8$–2.3 in steps of $\Delta n = 0.05$ and $Q = 4$–$28\ \mu$K in steps of $\Delta Q = 0.5\ \mu$K. A much finer resolution in Q and n was later obtained by two-dimensional interpolation of the above-defined grid of $L(Q, n)$-points. It was verified that with 5000 realizations the results converged to a stable value.

The bias and the errors on the estimated model parameters were obtained using the Monte Carlo procedure described above but with synthetic input data (for a fixed model) with known Q_{in} and n_{in}. For each input realization, one obtains a set of values $\{Q_{\max}, n_{\max}\}$ that maximizes the likelihood. The mean of the $\{Q_{\max}, n_{\max}\}$ points yields the bias, and their dispersion yields the actual errors. The n-parameter is biased upward by ≈ 0.03, and Q is biased in the opposite direction by ≈ 0.26. The 1 σ errors are $\delta n = 0.2$ and $\delta Q = 3.0\ \mu$K. This would give us the uncertainty due to "cosmic variance" alone. The error on the parameters due to instrumental noise was estimated and added in quadrature. The latter contribution to the error was computed following the Monte Carlo procedure explained above, but instead of generating the model-dependent C_l, we took a single realization of the C_l (which was fixed throughout the procedure), and to it we added realizations of the instrumental noise power spectrum. The noise coefficients, $C_{l,\text{noise}}$, are directly obtained from the harmonic coefficients of $A-B$ map combinations. Since $A-B$ noise maps do not require a Galactic cut, a straightforward harmonic fit is applicable.

The debiased results for which $L(Q, n)$ is maximum are $n = 1.52 \pm 0.4$ and $Q = 16.3 \pm 6.0\ \mu$K, which are consistent with WRI94. Figure 3 shows the angular power spectrum corresponding to this best-fit $P(k)$ and *COBE*'s data points. To give an idea of the goodness of fit, the χ^2 per degree of freedom at $L_{\max}$ is 30.612/14.

We repeated the analysis with the C_l-coefficients ($3 \leq l \leq 18$) from the 4 year *COBE* data given by Tegmark (1996). For these data, the maximum likelihood analysis yields $n = 1.22 \pm 0.3$ and $Q = 16.3 \pm 4.5$. The analysis also reveals that these parameters are anticorrelated. That is, values of Q and n that follow the relation $Q(n) = 19.9 \exp[0.756(1-n)]$ lie approximately inside the 2 σ contour level.

Our results for Q and n are consistent (within the error bars) with those obtained by the *COBE* group, which are summarized in Table 4 of WRI94 and in Table 2 of Bennett et al. (1996), and, depending on the analysis method or the way the data were prepared (e.g., which map combination, exclusion or not of the quadrupole term, beam shape filter), their results for n range from 1.02 ± 0.4 to $1.42^{+0.49}_{-0.55}$ for the 2 year results and from $1.23^{+0.23}_{-0.29}$ to $1.30^{+0.30}_{-0.34}$ for the 4 year results. One fact worth noting is that, independent of the n-values, a power-law form for the spectrum of PDFs is indeed consistent with the $P(k)$ obtained here directly from the CMB data without an a priori assumption about its shape.

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REFERENCES

Bennett, C. L., et al. 1996, ApJ, 464, L1
———. 1994, ApJ, 436, 423
Bond, J. R., & Efstathiou, G. 1987, MNRAS, 226, 655
Fabbri, R., Lucchin, F., & Matarrese, S. 1987, ApJ, 315, 1
Ganga, K., Cheng, E., Meyer, S., & Page, L. 1993, ApJ, 410, L57
Hancock, S., Davies, R. D., Lasenby, A. N., Gutiérrez de la Cruz, C. M., & Watson, R. A. 1994, Nature, 367, 333
Harrison, E. R. 1970, Phys. Rev. D, 1, 2726
Górski, K. M., Hinshaw, G., Banday, A. J., Bennett, C. L., Wright, E. L., Kogut, A., Smoot, G. F., & Lubin, P. 1994, ApJ, 430, L89
Sachs, K., & Wolfe, A. M. 1967, ApJ, 147, 73
Stromberg, K. R. 1981, Introduction to Classical Real Analysis (Belmont, CA: Wadsworth)
Tegmark, M. 1996, ApJ, 464, L35
Wright, E. L., Smoot, G. F., Bennett, C. L., & Lubin, P. M. 1994a, ApJ, 436, 443
Wright, E. L., Smoot, G. F., Kogut, A., Hinshaw, G., Tenorio, L., Lineweaver, C., Bennett, C. L., & Lubin, P. M. 1994b, ApJ, 420, 1
Zeldovich, Ya. B. 1972, MNRAS, 160, 1P